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1998 J. Phys. A: Math. Gen. 31 6839

(http://iopscience.iop.org/0305-4470/31/32/007)

J. Phys. A: Math. Gen. **31** (1998) 6839–6848. Printed in the UK PII: S0305-4470(98)93080-3

Two-paraboson coherent states

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Received 6 April 1998

Abstract. In this paper we introduce two-paraboson coherent states (TPCS) defined as eigenstates of a linear combination of parabose creation and annihilation operators. The wavefunctions of TPCS in various bases are explicitly calculated and the basic squeezing properties of these states are discussed.

1. Introduction

In recent years there has been increasing interest in various generalized statistics, which include parastatistics [1], anyon statistics [2], infinite statistics [3] and the statistics of quons [4] (particles whose creation and annihilation operators obey the q-deformed commutation relations). The main motivation comes from their potential applications in condensed matter physics, such as to the theory of fractional quantum Hall effect [5] and to the theory of anyon superconductivity [6]. Of these generalizations, parastatistics was first introduced by H S Green four decades ago. This generalization, carried out at the level of the algebra of creation and annihilation operators, involves trilinear commutation relations in place of the bilinear relations that characterize Bose and Fermi systems. In addition, states in a parastatistics theory belong to many-dimensional representations of permutation group, this contrasts with the cases of Bose and Fermi statistics in which only the one-dimensional representations occur. In fact, parastatistics is a perfectly consistent local quantum theory. All norms in parastatistics theory are positive, there are no negative probabilities.

In order to effectively develop the possible applications of paraststistics in condensed matter physics, it is necessary to know the character of paraststistics as much as possible. At the early days of parastatistics, the structure of Fock space [7] and the coherent state representation [10] for parasystems were extensively studied. A few years ago, the canonical partition function for a non-trivial parasystem, a parasystem with order two, was derived [8], and the corresponding results for any order were obtained only at two years ago [9].

To our knowledge, the paraboson coherent state was investigated many years ago [10], another important kind of non-classical state, however, squeezed state for parabosons has never been appeared in literature. In this present paper we construct the two-paraboson coherent state (TPCS) in section 2, which are defined as eigenstates of a linear combination of parabose creation and annihilation operators. We calculate the wavefunctions of TPCS in various bases in section 3 and discuss their basic squeezing properties in section 4.

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2. Two-paraboson coherent states

The Fock space of a parabose system of order p, where p is a non-negative integer, is characterized by the trilinear commutation relations (for the sake of simplicity, only one degree of freedom of paraboson is considered in this paper)

$$\left[a,\{a^\dagger,a\}\right]=2a \qquad [a,a^{\dagger 2}]=2a^\dagger \qquad [a,a^2]=0 \tag{1}$$

and the supplementary conditions

$$a|0\rangle=0 \qquad aa^\dagger|0\rangle=p|0\rangle \tag{2}$$

where $|0\rangle$ is a unique vaccum state of the Fock space. Consider a unitary operator

$$U_z=\exp\left(\tfrac{1}{2}za^2-\tfrac{1}{2}z^*a^{\dagger 2}\right) \tag{3}$$

where $z=r\mathrm{e}^{\mathrm{i}\varphi}$ is an arbitrary complex number. Inspection of the above operator shows that $U_z^\dagger=U_z^{-1}=U_{-z}$. Using the trilinear commutation relations (1) we can perform the following canonical transformation:

$$\begin{aligned} b&=U_zaU_z^\dagger=\mu a+\nu a^\dagger\\ b^\dagger&=U_za^\dagger U_z^\dagger=\mu^*a^\dagger+\nu^*a \end{aligned} \tag{4}$$

where $\mu=\mu^*=\cosh r$, $\nu=\mathrm{e}^{\mathrm{i}\varphi}\sinh r$, and it is obvious that $|\mu|^2-|\nu|^2=1$. Of course, the unitary transformation (4) ensures that the operators b and $b^\dagger$ satisfy the same form of trilinear commutation relations as that of a and $a^\dagger$.

Similar to the ordinary boson case [11], the TPCS $|\beta,z\rangle$ can be defined to be the eigenstates of the operator b, which is a linear combinaiton of the parabose creation and annihilation operators $a^\dagger$ and a, with eigenvalue β:

$$b|\beta,z\rangle=\beta|\beta,z\rangle \tag{5}$$

where β is an arbitrary complex number. From equation (4) we see that the TPCS $|\beta,z\rangle$ can be written as

$$|\beta,z\rangle=U_z|\beta\rangle \tag{6}$$

where $|\beta\rangle$ is the parabose coherent state defined by

$$|\beta\rangle=E(|\beta|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{[n]!}}|n\rangle \tag{7}$$

$|n\rangle$ being the number state of the parabose Fock space

$$|n\rangle=\frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle \tag{8}$$

and

$$[n]=n+\frac{p-1}{2}\left(1-(-)^n\right) \qquad E(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]!} \tag{9}$$

where $[n]!=[n][n-1]\cdots[1]$ and $[0]!\equiv 1$. When $p=1$, $[n]$ and $E(x)$ reduce to the ordinary integer n and the exponential function e^x respectively. Equation (6) shows that the state $|\beta,z\rangle$ involves two parameters β and z. When $z=0$, the state $|\beta,z\rangle$ becomes the parabose coherent state $|\beta\rangle$.

From [10] we know that the parabose coherent states satisfy the completeness relation

$$\int \mathrm{d}^2\beta\ \mu(|\beta|^2)|\beta\rangle\langle\beta|=1 \tag{10}$$

where $d^2\beta = dx\,dy$, x and y respectively being the real and imaginary parts of β, and the integration is performed over the whole complex β plane. The integration weight function $\mu(t)$ in (10) is defined by

$$\mu(t) = \frac{E(t)}{2\pi}\int_{-\infty}^{\infty} ds\ M(s)e^{-\mathrm{i}ts} \qquad M(s) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{[n]!}{n!}(\mathrm{i}s)^n. \tag{11}$$

Multiplying equation (10) by U_z on the left and by $U_z^\dagger$ on the right, we have

$$\int d^2\beta\ \mu(|\beta|^2)|\beta, z\rangle\langle\beta, z| = 1 \tag{12}$$

which means that the TPCS satisfy the same completeness relation and any state vector $|\psi\rangle$ can be expended in terms of $|\beta, z\rangle$. Furthermore, the TPCS have the same scalar product as the parabose coherent states

$$\langle\beta, z|\alpha, z\rangle = \frac{E(\beta^*\alpha)}{\sqrt{E(|\alpha|^2)E(|\beta|^2)}} \tag{13}$$

which implies that the TPCS are normalized.

3. Wavefunctions in various bases

First let us determine the wavefunction of the TPCS $|\beta, z\rangle$ in the parabose coherent state representation $\langle\alpha|\beta, z\rangle$.

We would like to point out that the parabose creation operator $a^\dagger$ acting on a parabose coherent state $|\alpha\rangle$ gives

$$a^\dagger|\alpha\rangle = \frac{\partial}{\partial\alpha}|\alpha\rangle + \left(\frac{\alpha^*}{2} + \frac{p-1}{4\alpha} + \frac{p-1}{4\alpha}E(|\alpha|^2)^{-1}E(-|\alpha|^2)\right)|\alpha\rangle - \frac{p-1}{2\alpha}|-\alpha\rangle \tag{14}$$

which reduces to the ordinary case when $p = 1$. Using this formula and (4), we have

$$\begin{aligned}\beta\langle\alpha|\beta, z\rangle &= \beta\langle\alpha|U_z|\beta\rangle = \langle\alpha|\left(a\cosh r + a^\dagger e^{-\mathrm{i}\varphi}\sinh r\right)U_z|\beta\rangle \\ &= \left[\cosh r\frac{\partial}{\partial\alpha^*} + \cosh r\left(\frac{\alpha}{2} + \frac{p-1}{4\alpha^*} + \frac{p-1}{4\alpha^*}E(|\alpha|^2)^{-1}E(-|\alpha|^2)\right)\right. \\ &\quad \left. + e^{-\mathrm{i}\varphi}\sinh r\,\alpha^*\right]\langle\alpha|\beta, z\rangle - \frac{p-1}{2\alpha^*}\cosh r\,\langle-\alpha|\beta, z\rangle.\end{aligned} \tag{15}$$

The solution of (15) is of the form

$$\langle\alpha|\beta, z\rangle = K(\alpha, \beta, \beta^*, r, \varphi)E(|\alpha|^2)^{-1/2}E\left(\frac{\alpha^*\beta}{\cosh r}\right)\exp\left(-e^{-\mathrm{i}\varphi}\tanh r\frac{\alpha^{*2}}{2}\right). \tag{16}$$

We can use the unitarity of U_z to determine the functional form of K. Substituting (16) in $\langle\alpha|U_z|\beta\rangle^* = \langle\beta|U_z^\dagger|\alpha\rangle = \langle\beta|U_{-z}|\alpha\rangle$, we have

$$\begin{aligned}&K^*(\alpha, \beta, \beta^*, r, \varphi)E(|\alpha|^2)^{-1/2}E\left(\frac{\alpha\beta^*}{\cosh r}\right)\exp\left(-e^{\mathrm{i}\varphi}\tanh r\frac{\alpha^2}{2}\right) \\ &\quad = K(\beta, \alpha, \alpha^*, r, \varphi+\pi)E(|\beta|^2)^{-1/2}E\left(\frac{\alpha\beta^*}{\cosh r}\right)\exp\left(e^{-\mathrm{i}\varphi}\tanh r\frac{\beta^{*2}}{2}\right).\end{aligned} \tag{17}$$

The solution to this functional equation for K is

$$K(\alpha, \beta, \beta^*, r, \varphi) = E(|\beta|^2)^{-1/2} \exp\left(\mathrm{e}^{\mathrm{i}\varphi} \tanh r \frac{\beta^2}{2}\right). \tag{18}$$

Thus $\langle\alpha|\beta, z\rangle$ is of the form

$$\langle\alpha|\beta, z\rangle = E(|\alpha|^2)^{-1/2} E(|\beta|^2)^{-1/2} E\left(\frac{\alpha^*\beta}{\cosh r}\right) \exp\left(\tanh r \frac{\mathrm{e}^{\mathrm{i}\varphi}\beta^2 - \mathrm{e}^{-\mathrm{i}\varphi}\alpha^{*2}}{2}\right). \tag{19}$$

The normalization condition

$$\int \mathrm{d}^2\alpha \; \mu(|\alpha|^2) \, |\langle\alpha|\beta, z\rangle|^2 = 1 \tag{20}$$

gives another constant factor $(\cosh r)^{-p/2}$ to the wavefunction $\langle\alpha|\beta, z\rangle$, so finally we have

$$\langle\alpha|\beta, z\rangle = (\cosh r)^{-p/2} E(|\alpha|^2)^{-1/2} E(|\beta|^2)^{-1/2} E\left(\frac{\alpha^*\beta}{\cosh r}\right) \exp\left(\tanh r \frac{\mathrm{e}^{\mathrm{i}\varphi}\beta^2 - \mathrm{e}^{-\mathrm{i}\varphi}\alpha^{*2}}{2}\right). \tag{21}$$

When $p = 1$, this $\langle\alpha|\beta, z\rangle$ will give the familiar wavefunction of two-photon coherent state in ordinary coherent state representation [11].

In fact, there is another simple way to get the wavefunction (21). To see this, let us introduce notations $K_\pm$ and K_0 defined by

$$K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \qquad K_0 = \tfrac{1}{4}\{a^\dagger, a\}. \tag{22}$$

From the trilinear commutation relations (1) it is clear that $K_\pm$ and K_0 satisfy the su$(1, 1)$ Lie algebra relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{23}$$

Using the disentangling theorem of su$(1, 1)$ [12], we have

$$U_z = \exp\left(-\mathrm{e}^{-\mathrm{i}\varphi} \tanh r \, \frac{a^{\dagger 2}}{2}\right) \exp\left(-\ln\cosh r \, \frac{\{a^*, a\}}{2}\right) \exp\left(\mathrm{e}^{\mathrm{i}\varphi} \tanh r \, \frac{a^2}{2}\right) \tag{24}$$

which leads to

$$\langle\alpha|\beta, z\rangle = \langle\alpha|U_z|\beta\rangle = \exp\left(\tanh r \frac{\mathrm{e}^{\mathrm{i}\varphi}\beta^2 - \mathrm{e}^{-\mathrm{i}\varphi}\alpha^{*2}}{2}\right) \langle\alpha| \exp\left(-\ln\cosh r \, \frac{\{a^*, a\}}{2}\right)|\beta\rangle. \tag{25}$$

Substituting the parabose coherent state expression (7) in (25) and recalling that

$$\{a^\dagger, a\}|n\rangle = \big([n] + [n+1]\big)|n\rangle = (2n + p)|n\rangle \tag{26}$$

we obtain

$$\begin{aligned}
\langle\alpha|\beta, z\rangle &= E(|\alpha|^2)^{-1/2} E(|\beta|^2)^{-1/2} \exp\left(\tanh r \frac{\mathrm{e}^{\mathrm{i}\varphi}\beta^2 - \mathrm{e}^{-\mathrm{i}\varphi}\alpha^{*2}}{2}\right) \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{[n]!} (\cosh r)^{-(2n+p)/2} \\
&= (\cosh r)^{-p/2} E(|\alpha|^2)^{-1/2} E(|\beta|^2)^{-1/2} E\left(\frac{\alpha^*\beta}{\cosh r}\right) \\
&\quad \times \exp\left(\tanh r \frac{\mathrm{e}^{\mathrm{i}\varphi}\beta^2 - \mathrm{e}^{-\mathrm{i}\varphi}\alpha^{*2}}{2}\right)
\end{aligned} \tag{27}$$

which coincides exactly with (21).

Now consider the wavefunction of the TPCS $|\beta, z\rangle$ in the parabose number representation $\langle n|\beta, z\rangle$. As a function of the variables x and t, $E(2xt)\exp(-t^2)$ can be expanded in a power series of t:

$$E(2xt)\exp(-t^2) = \sum_{n=0}^{\infty} \frac{H_n^{(p)}(x)}{[n]!} t^n \qquad |t| < \infty \tag{28}$$

where $H_n^{(p)}(x)$ is a deformation of the nth Hermite polynomial with argument x

$$H_n^{(p)}(x) = [n]! \sum_{l=0}^{[n/2]} \frac{(-)^n (2x)^{n-2l}}{l![n-2l]!} \tag{29}$$

where the notation $[k]$ on $\sum$ stands for the largest integer smaller than or equal to k. When $p = 1$, $H_n^{(p)}(x)$ becomes the ordinary Hermite polynomial. Using equation (28) we have

$$E\left(\frac{\alpha^*\beta}{\cosh r}\right)\exp\left(-\mathrm{e}^{-\mathrm{i}\varphi}\tanh r\frac{\alpha^{*2}}{2}\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}\mathrm{e}^{-\mathrm{i}\varphi}\tanh r)^{n/2}}{[n]!} H_n^{(p)}\left(\frac{\beta\mathrm{e}^{\mathrm{i}\varphi/2}}{\sqrt{\sinh 2r}}\alpha^{*n}\right). \tag{30}$$

Writing $\langle\alpha|\beta, z\rangle = \sum_n \langle\alpha|n\rangle\langle n|\beta, z\rangle$ and using equations (7) and (28), we find that

$$\langle n|\beta, z\rangle = (\cosh r)^{-p/2}([n]!)^{-1/2} E(|\beta|^2)^{-1/2} \left(\tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\varphi}\tanh r\right)^{n/2}$$
$$\times \exp\left(\mathrm{e}^{\mathrm{i}\varphi}\tanh r\frac{\beta^2}{2}\right) H_n^{(p)}\left(\frac{\beta\mathrm{e}^{\mathrm{i}\varphi/2}}{\sqrt{\sinh 2r}}\right). \tag{31}$$

When $z \to 0$ (i.e. $r \to 0$), the dominant term in $H_n^{(p)}$ is the first ($l = 0$) term, i.e.

$$H_n^{(p)}\left(\frac{\beta\mathrm{e}^{\mathrm{i}\varphi/2}}{\sqrt{\sinh 2r}}\right)\bigg|_{r\to 0} \to \left(\frac{2\beta\mathrm{e}^{\mathrm{i}\varphi/2}}{\sqrt{\sinh 2r}}\right)^n. \tag{32}$$

Thus from equation (31) we have

$$\langle n|\beta, z\rangle\big|_{z\to 0} = E(|\beta|^2)^{-1/2}\frac{\beta^n}{\sqrt{[n]!}} \tag{33}$$

which agrees with (7).

Let N_b be the 'quasiparaboson' number operator

$$N_b = \tfrac{1}{2}(b^\dagger b + bb^\dagger) - \tfrac{1}{2}p = U_z\left(\tfrac{1}{2}\{a^\dagger, a\} - \tfrac{1}{2}p\right)U_z^\dagger. \tag{34}$$

Then N_b has discrete positive eigenvalues n with ground state $|0_b\rangle$:

$$N_b|n_b\rangle = n|n_b\rangle \qquad |n_b\rangle = U_z|n\rangle \qquad N_b|0_b\rangle = 0. \tag{35}$$

Similarly to $|n\rangle$, the states $|n_b\rangle$ can be expressed in the form

$$|n_b\rangle = \frac{(b^\dagger)^n}{\sqrt{[n]!}}|0_b\rangle. \tag{36}$$

They are complete orthonormal. The operator b acts as the lowering operator for $|n_b\rangle$. In this basis the TPCS $|\beta, z\rangle$ are given by the simple expression

$$\langle n_b|\beta, z\rangle = \langle n|\beta\rangle = E(|\beta|^2)^{-1/2}\frac{\beta^n}{\sqrt{[n]!}}. \tag{37}$$

To conclude this section let us calculate the scalar product of two states $|\alpha, z\rangle$ and $|\beta, z'\rangle$, where

$$|\alpha, z\rangle = U_z|\alpha\rangle \qquad |\beta, z'\rangle = U_{z'}|\beta\rangle \tag{38}$$

($z = r\mathrm{e}^{\mathrm{i}\varphi},\ z' = r'\mathrm{e}^{\mathrm{i}\varphi'}$), which are generated from the parabose coherent states $|\alpha\rangle$ and $|\beta\rangle$ via different unitary transformations $U_z = \exp\left(\frac{1}{2}za^2 - \frac{1}{2}z^*a^{\dagger 2}\right)$ and $U_{z'} = \exp\left(\frac{1}{2}z'a^2 - \frac{1}{2}z'^*a^{\dagger 2}\right)$, respectively. By virtue of (24), we have

$$U_z^\dagger U_{z'} = \exp(\mathrm{e}^{-\mathrm{i}\varphi}\tanh r K_+)\exp(-2\ln\cosh r K_0)\exp(-\mathrm{e}^{\mathrm{i}\varphi}\tanh r K_-) \\ \times\exp(-\mathrm{e}^{-\mathrm{i}\varphi'}\tanh r' K_+)\exp(-2\ln\cosh r' K_0)\exp(\mathrm{e}^{\mathrm{i}\varphi'}\tanh r' K_-) \tag{39}$$

which leads to

$$\langle\alpha, z|\beta, z'\rangle = \exp\left(\mathrm{e}^{-\mathrm{i}\varphi}\tanh r\frac{\alpha^{*2}}{2} + \mathrm{e}^{\mathrm{i}\varphi'}\tanh r'\frac{\beta^2}{2}\right) \\ \times\langle\alpha|\exp(-2\ln\cosh r\ K_0)\exp(-\mathrm{e}^{\mathrm{i}\varphi}\tanh r\ K_-) \\ \times\exp(-\mathrm{e}^{-\mathrm{i}\varphi'}\tanh r'\ K_+)\exp(-2\ln\cosh r'\ K_0)|\beta\rangle. \tag{40}$$

Noting

$$\exp\left(-\ln\cosh r\frac{\{a^\dagger, a\}}{2}\right)|\alpha\rangle = (\cosh r)^{-p/2}E(|\alpha|^2)^{-1/2}E\left(\left|\frac{\alpha}{\cosh r}\right|^2\right)^{1/2}\left|\frac{\alpha}{\cosh r}\right\rangle \tag{41}$$

and using the formula (see the appendix)

$$\exp(\tau\ K_-)\exp(\lambda\ K_+) = \exp\left(\frac{\lambda}{1-\lambda\tau}\ K_+\right)\exp\left(-2\ln(1-\lambda\tau)\ K_0\right)\exp\left(\frac{\tau}{1-\lambda\tau}\ K_-\right) \tag{42}$$

we finally obtain

$$\langle\alpha, z|\beta, z'\rangle = \left(\cosh r\cosh r' - \mathrm{e}^{\mathrm{i}(\varphi-\varphi')}\sinh r\sinh r'\right)^{-p/2}E(|\alpha|^2)^{-1/2}E(|\beta|^2)^{-1/2} \\ \times E\left(\frac{\alpha^*\beta}{\cosh r\cosh r' - \sinh r\sinh r'\exp\left(\mathrm{i}(\varphi-\varphi')\right)}\right) \\ \times\ \exp\left(\frac{\mathrm{e}^{-\mathrm{i}\varphi}\cosh r'\sinh r - \mathrm{e}^{-\mathrm{i}\varphi'}\cosh r\sinh r'}{\cosh r\cosh r' - \sinh r\sinh r'\exp\left(\mathrm{i}(\varphi-\varphi')\right)}\frac{\alpha^{*2}}{2}\right. \\ \left.-\frac{\mathrm{e}^{\mathrm{i}\varphi}\cosh r'\sinh r - \mathrm{e}^{\mathrm{i}\varphi'}\cosh r\sinh r'}{\cosh r\cosh r' - \sinh r\sinh r'\exp\left(\mathrm{i}(\varphi-\varphi')\right)}\frac{\beta^2}{2}\right). \tag{43}$$

Similarly we also have

$$\langle n|m_b\rangle = \langle n|U_z|m\rangle \\ = (\cosh r)^{-(2m+p)/2}([n]![m]!)^{1/2} \\ \times\sum_{k=0}^{[n/2]}\sum_{l=0}^{[m/2]}\frac{(-)^k(\frac{1}{2}\tanh r)^{l+k}\mathrm{e}^{\mathrm{i}(l-k)\varphi}(\cosh r)^{2l}}{k!\,l!\sqrt{[n-2k]!\,[m-2l]!}}\delta_{n-2k,m-2l} \tag{44}$$

which shows that n and m must be both even or both odd integers in order for $\langle n|m_b\rangle$ to be non-vanishing. Thus only an even number of parabosons would be counted for the state $|0_b\rangle = |0, z\rangle$.

4. Uncertainty relations and squeezing properties

In this section, we discuss some properties of the TPCS defined in section 2. Introducing Hermitian operators X and P defined by

$$X = \frac{a + a^\dagger}{\sqrt{2}} \qquad P = \frac{a - a^\dagger}{\sqrt{2}\mathrm{i}} \tag{45}$$

we see that the commutator of X and P is given by $[X, P] = \mathrm{i}[a, a^\dagger]$, and the Hamiltonian of a free parabose sustem with a single degree of freedom can be written as

$$H = \tfrac{1}{2}(a^\dagger a + aa^\dagger) = \tfrac{1}{2}(X^2 + P^2). \tag{46}$$

The expectation values of the operators X and P in the TPCS can easily be calculated:

$$\begin{aligned}\langle X\rangle &= \langle\beta, z|X|\beta, z\rangle = \frac{1}{\sqrt{2}}\left((\cosh r - \mathrm{e}^{\mathrm{i}\varphi}\sinh r)\beta + (\cosh r - \mathrm{e}^{-\mathrm{i}\varphi}\sinh r)\beta^*\right)\\ \langle P\rangle &= \langle\beta, z|P|\beta, z\rangle = \frac{1}{\sqrt{2}\mathrm{i}}\left((\cosh r + \mathrm{e}^{\mathrm{i}\varphi}\sinh r)\beta - (\cosh r + \mathrm{e}^{-\mathrm{i}\varphi}\sinh r)\beta^*\right).\end{aligned} \tag{47}$$

Similarly, we have

$$\begin{aligned}\langle H\rangle &= \langle\beta, z|H|\beta, z\rangle\\ &= \cosh 2r\left(|\beta|^2 + \frac{1}{2} + \frac{p-1}{2}E(|\beta|^2)^{-1}E(-|\beta|^2)\right)\\ &\quad - \sinh 2r\left(\tfrac{1}{2}\beta^2\mathrm{e}^{\mathrm{i}\varphi} + \tfrac{1}{2}\beta^{*2}\mathrm{e}^{-\mathrm{i}\varphi}\right).\end{aligned} \tag{48}$$

Noting that $X^2 = H + \frac{1}{2}(a^2 + a^{\dagger 2})$ and $P^2 = H - \frac{1}{2}(a^2 + a^{\dagger 2})$, we can write

$$\begin{aligned}\langle X^2\rangle &= \langle\beta, z|X^2|\beta, z\rangle\\ &= \tfrac{1}{2}\left((\cosh r - \sinh r\,\mathrm{e}^{\mathrm{i}\varphi})\beta + (\cosh r - \sinh r\,\mathrm{e}^{-\mathrm{i}\varphi})\beta^*\right)^2\\ &\quad + \tfrac{1}{2}(\cosh 2r - \sinh 2r\cos\varphi)\left(1 + (p-1)E(|\beta|^2)^{-1}E(-|\beta|^2)\right)\\ \langle P^2\rangle &= \langle\beta, z|P^2|\beta, z\rangle\\ &= -\tfrac{1}{2}\left((\cosh r + \sinh r\,\mathrm{e}^{\mathrm{i}\varphi})\beta - (\cosh r + \sinh r\,\mathrm{e}^{-\mathrm{i}\varphi})\beta^*\right)^2\\ &\quad + \tfrac{1}{2}(\cosh 2r + \sinh 2r\cos\varphi)\left(1 + (p-1)E(|\beta|^2)^{-1}E(-|\beta|^2)\right).\end{aligned} \tag{49}$$

Thus the variances of the operators X and P in the TPCS are of the form

$$\begin{aligned}\langle(\triangle X)^2\rangle &\equiv \langle X^2\rangle - \langle X\rangle^2\\ &= \tfrac{1}{2}(\cosh 2r - \sinh 2r\cos\varphi)\left(1 + (p-1)E(|\beta|^2)^{-1}E(-|\beta|^2)\right)\\ \langle(\triangle P)^2\rangle &\equiv \langle P^2\rangle - \langle P\rangle^2\\ &= \tfrac{1}{2}(\cosh 2r + \sinh 2r\cos\varphi)\left(1 + (p-1)E(|\beta|^2)^{-1}E(-|\beta|^2)\right)\end{aligned} \tag{50}$$

which lead to

$$\langle(\triangle X)^2\rangle\langle(\triangle P)^2\rangle = \tfrac{1}{4}\left(1 + (\sinh 2r)^2(\sin\varphi)^2\right)\left(1 + (p-1)E(|\beta|^2)^{-1}E(-|\beta|^2)\right)^2. \tag{51}$$

Using equation (4) we have $\langle\beta, z|[a, a^\dagger]|\beta, z\rangle = \langle\beta|[a, a^\dagger]|\beta\rangle$. Noting that $-\mathrm{i}\langle[X, P]\rangle = \langle[a, a^\dagger]\rangle = \langle\beta, z|[a, a^\dagger]|\beta, z\rangle$ and comparing [10] with

$$\langle\beta|[a, a^\dagger]|\beta\rangle = 1 + (p-1)E(|\beta|^2)^{-1}E(-|\beta|^2) \tag{52}$$

we find that

$$\langle(\triangle X)^2\rangle\langle(\triangle P)^2\rangle = \frac{1}{4}\left(1+(\sinh 2r)^2(\sin\varphi)^2\right)|\langle[X,P]\rangle|^2 \geqslant \frac{1}{4}|\langle[X,P]\rangle|^2 \tag{53}$$

which shows that for TPCS, when z is real ($\varphi = 0$), the uncertainty relation reduces to an equality. However, since $[X, P]$ is in general not a c-number, the right-hand side of (53) itself depends on the given state. Hence the TPCS are not the minimum uncertainty states in the absolute sense (except for the $p = 1$ case). On the other hand, since the variance of the operators X and P in the parabose coherent state $|\beta\rangle$ is [10]

$$\langle\beta|(\triangle X)^2|\beta\rangle = \langle\beta|(\triangle P)^2|\beta\rangle = \frac{1}{2}\left(1+(p-1)E(|\beta|^2)^{-1}E(-|\beta|^2)\right) \tag{54}$$

it is obvious from (50) that

$$\begin{aligned}\langle\beta,z|(\triangle X)^2|\beta,z\rangle &= (\cosh 2r - \sinh 2r\cos\varphi)\,\langle\beta|(\triangle X)^2|\beta\rangle\\ \langle\beta,z|(\triangle P)^2|\beta,z\rangle &= (\cosh 2r + \sinh 2r\cos\varphi)\,\langle\beta|(\triangle P)^2|\beta\rangle\end{aligned} \tag{55}$$

which mean that the case

$$\langle\beta,z|(\triangle X)^2|\beta,z\rangle \leqslant \langle\beta|(\triangle X)^2|\beta\rangle \qquad \text{or} \qquad \langle\beta,z|(\triangle P)^2|\beta,z\rangle \leqslant \langle\beta|(\triangle P)^2|\beta\rangle \tag{56}$$

may occur for some ranges of the parameter z and the TPCS may exhibit 'squeezing' effects for these cases. It is only in this sense that we also call $|\beta, z\rangle$ the parabose squeezed state.

Finally, we consider the time evolution for the case of a free parabose oscillator initially in the state $|\beta, z\rangle$. In the Schrödinger representation, the state at time t which evolves from $|\beta, z\rangle$ at $t = 0$ is given by $|\beta, z; t\rangle = \exp(-\mathrm{i}Ht)|\beta, z\rangle$. H is the Hamiltonian operator $H = \frac{1}{2}\omega\{a^\dagger, a\}$ governing the system involved. Thus we have

$$\mathrm{i}\frac{\partial}{\partial t}|\beta,z;t\rangle = H\,|\beta,z;t\rangle \tag{57}$$

and

$$\begin{aligned}|\beta,z;t\rangle &= \mathrm{e}^{-\mathrm{i}Ht}|\beta,z\rangle\\ &= \exp\left(-\tfrac{1}{2}\mathrm{i}\omega t\{a^\dagger,a\}\right)U_z\exp\left(\tfrac{1}{2}\mathrm{i}\omega t\{a^\dagger,a\}\right)\exp\left(-\tfrac{1}{2}\mathrm{i}\omega t\{a^\dagger,a\}\right)|\beta\rangle\\ &= \exp\left(\tfrac{1}{2}z\mathrm{e}^{2\mathrm{i}\omega t}a^2 - \tfrac{1}{2}z^*\mathrm{e}^{-2\mathrm{i}\omega t}a^{\dagger 2}\right)\mathrm{e}^{-\mathrm{i}p\omega t/2}|\beta\mathrm{e}^{-\mathrm{i}\omega t}\rangle\\ &= \mathrm{e}^{-\mathrm{i}p\omega t/2}U_{z\mathrm{e}^{2\mathrm{i}\omega t}}|\beta\mathrm{e}^{-\mathrm{i}\omega t}\rangle\end{aligned} \tag{58}$$

where the relation $\exp\left(-\frac{1}{2}\mathrm{i}\omega t\{a^\dagger,a\}\right)|\beta\rangle = \mathrm{e}^{-\mathrm{i}p\omega t/2}|\beta\mathrm{e}^{-\mathrm{i}\omega t}\rangle$ has been used. From equation (58) we see that $|\beta, z; t\rangle$ is still a two-paraboson coherent state with time-dependent parameters $z\mathrm{e}^{2i\omega t}$ and $\beta\mathrm{e}^{-\mathrm{i}\omega t}$ in place of the initially time-independent ones z and β.

Note added in proof. The author would like to thank one of the referees for drawing to his attention a paper by Bagchi and Bhaumik [13], in which a similar topic was discussed in different way.

Acknowledgments

The author wishes to thank Professor C N Yang for his kindly hospitality at the Institute for Theoretical Physics, State University of New York at Stony Brook where the work was done. This work is supported partly by the National Natural Science Foundation of China.

Appendix A. Proof of equation (42)

Using equations (23) it can easily be seen that

$$\exp(-\lambda K_+)K_- \exp(\lambda K_+) = K_- + 2\lambda K_0 + \lambda^2 K_+ \tag{A1}$$

which leads to

$$\exp(\tau K_-)\exp(\lambda K_+) = \exp(\lambda K_+)\exp(\lambda^2\tau K_+ + 2\lambda\tau K_0 + \tau K_-). \tag{A2}$$

Let us introduce a real parameter t and write

$$\exp\left(t\,(\lambda^2\tau K_+ + 2\lambda\tau K_0 + \tau K_-)\right) = \exp(p_+(t)\,K_+)\exp(p_0(t)\,K_0)\exp(p_-(t)\,K_-) \tag{A3}$$

where $p_+(t)$, $p_0(t)$ and $p_-(t)$ are functions to be determined which subject to the constraints $p_+(0) = p_0(0) = p_-(0) = 0$. Differentiating (A3) with respect to t, we have

$$\begin{aligned}(\lambda^2\tau K_+ &+ 2\lambda\tau K_0 + \tau K_-)\exp\left(t\,(\lambda^2\tau K_+ + 2\lambda\tau K_0 + \tau K_-)\right)\\ &= p'_+ K_+ \exp(p_+\,K_+)\exp(p_0\,K_0)\exp(p_-\,K_-)\\ &\quad + p'_0\exp(p_+\,K_+)K_0\exp(p_0\,K_0)\exp(p_-\,K_-)\\ &\quad + p'_-\exp(p_+\,K_+)\exp(p_0\,K_0)K_-\exp(p_-\,K_-)\end{aligned} \tag{A4}$$

where the primes indicate differentiation with respect to t. Multiplying from the right by

$$\exp\left(-t\,(\lambda^2\tau K_+ + 2\lambda\tau K_0 + \tau K_-)\right) = \exp(-p_-(t)\,K_-)\exp(-p_0(t)\,K_0)\exp(-p_+(t)\,K_+) \tag{A5}$$

we obtain

$$\begin{aligned}\lambda^2\tau K_+ &+ 2\lambda\tau K_0 + \tau K_-\\ &= (p'_+ - p'_0 p_+ + p_+^2 p'_- \mathrm{e}^{-p_0})K_+ + (p'_0 - 2p_+p'_-\mathrm{e}^{-p_0})K_0 + p'_-\mathrm{e}^{-p_0}K_-\end{aligned} \tag{A6}$$

where the commutation relation (23) of su(1, 1) are used. We identify the coefficients of the respective basis elements of the su(1, 1) Lie algebra and obtain a system of coupled nonlinear equations:

$$\begin{aligned}&p'_-\,\mathrm{e}^{-p_0} = \tau\\ &p'_0 - 2\,p_+\,p'_-\,\mathrm{e}^{-p_0} = 2\,\lambda\,\tau\\ &p'_+ - p'_0\,p_+ + p_+^2\,p'_-\,\mathrm{e}^{-p_0} = \lambda^2\,\tau\end{aligned} \tag{A7}$$

with the initial conditions $p_+(0) = p_0(0) = p_-(0) = 0$. Eliminating e^{-p_0} from these three equations, we obtain

$$p'_0 = \tau\,p_+ + 2\,\lambda\,\tau \qquad p'_+ - 2\,\lambda\,\tau\,p_+ - \tau\,p_+^2 = \lambda^2\,\tau. \tag{A8}$$

Making the substitution $p_+ = y/\tau$, $y(0) = 0$, followed by $y = -u'/u$, $u'(0) = 0$, $u(0) = 1$, we transform the last equation of (A8) into the second-order, ordinary differential equation

$$u'' - 2\,\lambda\,\tau u' + \lambda^2\,\tau^2\,u = 0 \tag{A9}$$

with constant coefficients. Its solution is

$$u = \mathrm{e}^{\lambda\,\tau\,t} - \lambda\,\tau\,t\,\mathrm{e}^{\lambda\,\tau\,t} \tag{A10}$$

which leads to $y = (\lambda^2\,\tau^2\,t)/(1 - \lambda\,\tau\,t)$ and further to

$$p_+ = \frac{\lambda^2\,\tau\,t}{1 - \lambda\,\tau\,t} \qquad p_0 = -2\ln(1 - \lambda\,\tau\,t) \qquad p_- = \frac{\tau\,t}{1 - \lambda\,\tau\,t}. \tag{A11}$$

Thus when $t = 1$ equation (A3) becomes

$$\exp(\lambda^2\,\tau\,K_+ + 2\,\lambda\,\tau\,K_0 + \tau\,K_-)$$
$$= \exp\left(\frac{\lambda^2\,\tau}{1-\lambda\,\tau}K_+\right)\exp(-2\,\ln(1-\lambda\,\tau)\,K_0)\exp\left(\frac{\tau}{1-\lambda\,\tau}K_-\right). \tag{A12}$$

Substituting this expression in (A2) we arrive at equation (42).

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